ON SOLVING THE IRREGULARLY-SHAPED PLATE BUCKLING PROBLEM
https://sci-hub.st/10.2514/6.1993-14

## Summary :

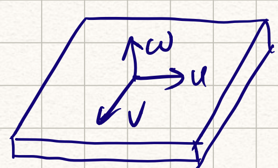
| | Strain Energy U | Virtual work Energy T |
|---|---|---|
| Plane stress | | |
| Buckling (Bending) | | |

$$\begin{cases} \Pi = U - T \\ \bar{\Pi} = \Pi + \lambda_i R_i \end{cases}$$

Subparametric mapping $(x, y) \rightarrow (\xi, \eta)$

Plane stress : 2-D in-plane deformation $u, v$

Buckling stress : 3-D out-of-plane deformation $w$



# Steps :

— Use plane stress to get stress distribution

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [\sigma] = [G]\{\varepsilon\}$$

— Get load distributions

$$\begin{cases} N_x = \sigma_{xx} h \\ N_y = \sigma_{yy} h \\ N_{xy} = \sigma_{xy} h \end{cases}$$

— Use bending (buckling) min potential Energy to get lateral displacement $\omega$.

# Irregular-Shaped Plate Buckling

Principle : minimum potential energy

Method : provide a general displacement function with unknown coefficients and without regard to the element's nodes.

Augmented : $\overline{\Pi} = \Pi + \lambda_i R_i$

Lagrange multiplier | constraint conditions

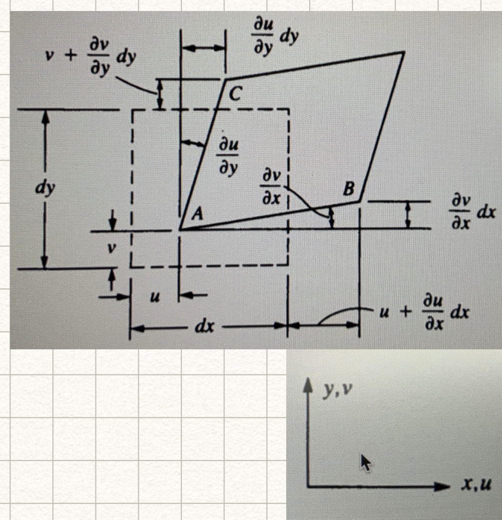
Potential Energy : $\Pi = U - T = e$

$U$ : strain energy

$T$ : virtual work done by the loads on the plate

$e$ : error associated with the chosen displacement function

Strain Energy : $U = \frac{1}{2}\int_V \{\varepsilon\}^T [\sigma]\, dV$




Plane stress : $\{\varepsilon\}^T = \{\varepsilon_x \;\; \varepsilon_y \;\; \gamma_{xy}\}$

$$[G] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

$[\sigma] = [G]\{\varepsilon\}$

↳ constitutive matrix

constant thickness $h$ → $U = \frac{Eh}{2(1-\nu^2)}\int_A \left(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y + \frac{(1-\nu)}{2}\gamma_{xy}^2\right) dA$

$\left.\begin{matrix} u \\ v \end{matrix}\right\}$ displacement

$u$ & $v$ : two independent displacement

For plane stress : $\varepsilon_x = \frac{\partial u}{\partial x}$ , $\varepsilon_y = \frac{\partial v}{\partial y}$ and $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

$$\Rightarrow U = \frac{Eh}{2(1-\nu^2)}\int_A \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\nu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{(1-\nu)}{2}\left(\frac{\partial u}{\partial y}\right)^2 + (1-\nu)\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{(1-\nu)}{2}\left(\frac{\partial v}{\partial x}\right)^2\right] dA$$

⇒ plane stress $U$

$\boxed{\text{We still need } T}$

In the plane stress case, two displacement functions are used, $u$ and $v$ with coefficients $a_{ij}$ and $b_{ij}$, respectively.

To minimize $\Pi$ with respect to $a_{ij}$ and $b_{ij}$:

$$d\Pi = \left(\frac{\partial U}{\partial a_{ij}} - \frac{\partial T}{\partial a_{ij}}\right) da_{ij} + \left(\frac{\partial U}{\partial b_{ij}} - \frac{\partial T}{\partial b_{ij}}\right) db_{ij} = 0$$

For plane stress

$$\left.\begin{aligned} [K] &\sim \frac{\partial U}{\partial a_{ij}} \ \& \ \frac{\partial U}{\partial b_{ij}} \\ [F] &\sim \frac{\partial T}{\partial a_{ij}} \ \& \ \frac{\partial T}{\partial b_{ij}} \\ \{x\} &\sim a_{ij} \ \& \ b_{ij} \end{aligned}\right\} \Rightarrow \underbrace{[K]}_{\text{Minimization of } U}\{x\} = \underbrace{[F]}_{\text{Minimization of } T}$$

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For bending (buckling), single displacement function $w$

$$\left([K_B] - N_{cr}\underbrace{[K_D]}_{\text{Minimization of } T}\right)\{x\} = 0$$

Some multiplier of the combination of $N_x, N_y, N_{xy}$

For Bending (Buckling)

minimization of

$$U = \frac{D}{2}\int_A \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dA$$

$w$ is in terms of $c_{ij}$

Buckling / Bending $U$

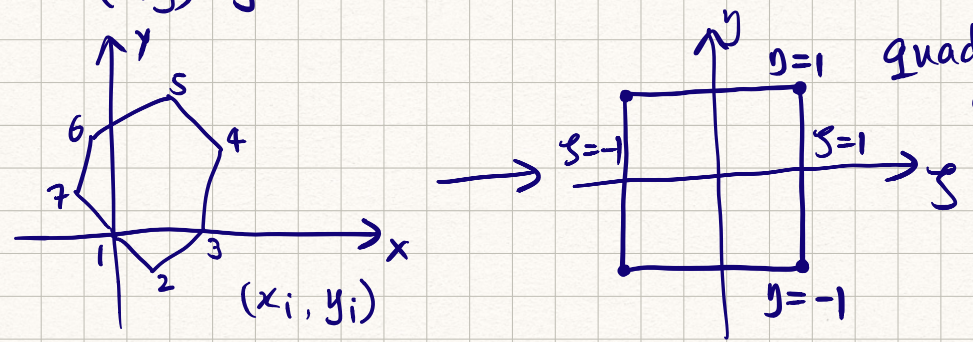
$$D = \frac{Eh^3}{12(1-\nu^2)}$$

→ strain energy for plane bending (buckling)

# Subparametric Mapping to a new domain

New domain : a square for 2D analysis.

E.g. eight-noded shape in a $(x, y)$ system $\xrightarrow{\text{mapping}}$ $(\xi, \eta)$ domain

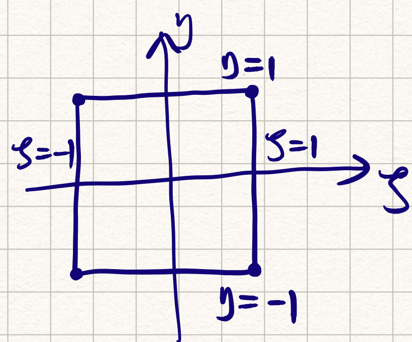



quadrilateral element

This mapping is done to easily integrate the strain energy and the work of the loads.

↓

Using Gauss integration




Method:

Using 8 nodes mapping,

$\Rightarrow$ $x$ & $y$ both contain 8 terms:

$$x = C_1\xi^2\eta + C_2\xi^2 + C_3\xi + C_4\xi\eta^2 + C_5\eta^2 + C_6\eta + C_7\xi\eta + C_8$$

$$y = D_1\xi^2\eta + D_2\xi^2 + D_3\xi + D_4\xi\eta^2 + D_5\eta^2 + D_6\eta + D_7\xi\eta + D_8$$

Using the eight point $(x_i, y_i)$,

coefficients $C_i$ and $D_i$ can be found via

$$C_1 = \frac{1}{4}(-x_1 + 2x_2 - x_3 + x_5 - 2x_6 + x_7),$$

$$C_2 = \frac{1}{4}(x_1 - 2x_2 + x_3 + x_5 - 2x_6 + x_7),$$

$$C_3 = \frac{1}{4}(2x_4 - 2x_8),$$

$$C_4 = \frac{1}{4}(-x_1 + x_3 - 2x_4 + x_5 - x_7 + 2x_8),$$

$$C_5 = \frac{1}{4}(x_1 + x_3 - 2x_4 + x_5 - x_7 - 2x_8),$$

$$C_6 = \frac{1}{4}(-2x_2 + 2x_6),$$

$$C_7 = \frac{1}{4}(x_1 - x_3 + x_5 - x_7),$$

$$C_8 = \frac{1}{4}(-x_1 + 2x_2 - x_3 + 2x_4 - x_5 + 2x_6 - x_7 + 2x_8).$$

Strain Energy in $(\xi, \eta)$ domain

constant

$$U = \frac{1}{2}\int_V \{\varepsilon\}^T[\sigma]\,dV = \frac{1}{2}\int_{\hat{V}} \{\varepsilon\}^T[\sigma]\cdot|J|\,d\hat{V} = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} h\cdot\{\varepsilon\}^T[\sigma]\,|J|\,d\xi\,d\eta$$

$|J|$ is the Jacobian, $|J| = \frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi}$

$d\hat{V} = d\xi\, d\eta\, dh$

$\xi$ and $\eta$ both have limits of $-1$ to $1$

$$\boxed{U = \frac{h}{2}\int_{-1}^{1}\int_{-1}^{1} \{\varepsilon\}^T\cdot[\sigma]\cdot|J|\,d\xi\,d\eta}$$

# Plane Stress Analysis ⟸ Not Buckling

- The displacement functions will be based in the parametric space $(\xi, \eta)$, and are general polynomials.
- The displacement functions:

$$u = \sum_i \sum_j a_{ij} \xi^i \eta^j$$

$$v = \sum_i \sum_j b_{ij} \xi^i \eta^j$$

Q: what is $i$ and $j$ ?

$\xi$ & $\eta$ are functions of $x$ and $y$

- By writing the displacement functions in this way, BCs on the sides $\xi = \pm 1$ and $\eta = \pm 1$ are easily applied.

- Boundary conditions:

For example: no displacement on $\xi = -1$

$$u = \sum_j \eta^j \sum_i a_{ij} (-1)^i = 0$$

$$v = \sum_j \eta^j \sum_i b_{ij} (-1)^i = 0$$

These equations make up $R_i$ in $\overline{\Pi} = \Pi + \lambda_i R_i$

- Strain Energy in $(\xi, \eta)$ domain

$$U = \frac{Eh}{2(1-\nu^2)} \int_{-1}^{1} \int_{-1}^{1} \Big[ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\nu \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{(1-\nu)}{2} \left(\frac{\partial u}{\partial y}\right)^2 + (1-\nu) \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{(1-\nu)}{2} \left(\frac{\partial v}{\partial x}\right)^2 \Big] |J| \, d\xi \, d\eta$$

# Virtual Work (Plane Stress)

— Assume : distributed load on edges.

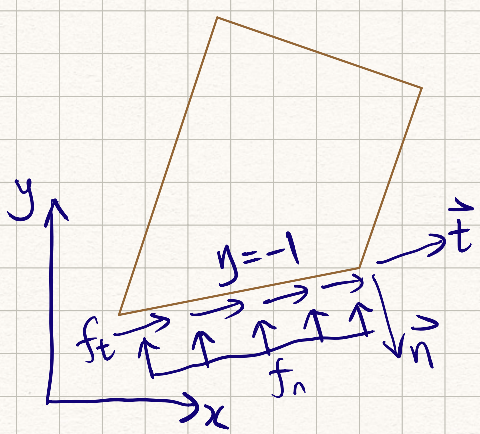
— The method is compliant to a variety of loading conditions.

— The loads are defined as normal and tangential tractions on the edges.

— For plane stress, the virtual work is a surface integral

$$T = \int_S \vec{F_s} \cdot \vec{u} \, dS$$

$\vec{F_s}$ : force per unit length




$$\vec{F_s} = f_t \cdot \vec{t} - f_n \cdot \vec{n}$$

in $(x, y)$

— The transformation from $(\vec{n}, \vec{t})$ to $(\vec{i}, \vec{j})$

$$\begin{Bmatrix} \vec{t} \\ \vec{n} \end{Bmatrix} = \frac{1}{|t|} \begin{bmatrix} t_x & t_y \\ t_y & -t_x \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}$$

$$\vec{F_s} = \frac{1}{|t|} \left[ (-f_n t_y + f_t t_x) \vec{i} + (f_n t_x + f_t t_y) \vec{j} \right]$$

$$\vec{u} = u\vec{i} + v\vec{j}$$

$$\vec{s} = x\vec{i} + y\vec{j} \quad , \quad d\vec{s} = dx\vec{i} + dy\vec{j}$$

$$\vec{t} = \frac{1}{|t|}(t_x \vec{i} + t_y \vec{j}) \Rightarrow \text{in } (x, y)$$

$$ds = \vec{t} \, d\vec{s} \quad ,$$

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta$$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$$

plane stress

$\Downarrow$

$\Rightarrow$ The virtual work

$$T = \int_S \frac{1}{|t|^2}\left\{\left[(-f_n t_y + f_t t_x)u + (f_n t_x + f_t t_y)v\right]\cdot\left[\left(t_x\frac{\partial x}{\partial \xi} + t_y\frac{\partial y}{\partial \xi}\right)d\xi + \left(t_x\frac{\partial x}{\partial \eta} + t_y\frac{\partial y}{\partial \eta}\right)d\eta\right]\right.$$

$\Downarrow$ $\int_{-1}^{1}\int_{-1}^{1}$

(Repeated)

To minimize $\Pi$ with respect to $a_{ij}$ and $b_{ij}$ :

$$d\Pi = \left(\frac{\partial U}{\partial a_{ij}} - \frac{\partial T}{\partial a_{ij}}\right)da_{ij} + \left(\frac{\partial U}{\partial b_{ij}} - \frac{\partial T}{\partial b_{ij}}\right)db_{ij} = 0$$

$$\left.\begin{aligned} [K] &\sim \frac{\partial U}{\partial a_{ij}} \ \& \ \frac{\partial U}{\partial b_{ij}} \\ [F] &\sim \frac{\partial T}{\partial a_{ij}} \ \& \ \frac{\partial T}{\partial b_{ij}} \\ \{x\} &\sim a_{ij} \ \& \ b_{ij} \end{aligned}\right\} \Rightarrow \underbrace{[K]}_{\text{Minimization of } U}\{x\} = \underbrace{[F]}_{\text{Minimization of } T}$$

plane stress

$\Rightarrow$ + constraints R

$$\begin{bmatrix} K & R^T \\ R & 0 \end{bmatrix}\begin{Bmatrix} x \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} \Rightarrow \text{eigenvalue problem}$$

$\rightarrow$ coefficient of $u$ and $v$

$\rightarrow$ stress & strain throughout the plate.

# General Buckling

## Virtual work

$$T = \frac{1}{2}\int_A \left[ N_x \left(\frac{\partial w}{\partial x}\right)^2 + N_y \left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right] dA$$

— the parameters $N_x$, $N_y$ and $N_{xy}$ have an unknown magnitude which is determined in the eigenvalue problem

$$\left( [K_B] - N_{cr}[K_D] \right)\{x\} = 0$$

— The plane stress solution gives the certain stress distribution in to the above T equation.

$\Longrightarrow$ The plane stress solution gives the stress distribution on each edge / within the plate.

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In $(\xi, \eta)$ domain:

$$T = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} \left[ N_x \left(\frac{\partial w}{\partial x}\right)^2 + N_y \left(\frac{\partial w}{\partial y}\right)^2 + 2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right] \cdot |J(\xi,\eta)|\, d\xi\, d\eta$$

— T is evaluated in $(\xi, \eta)$ domain using Gaussian integration, with $N_x$, $N_y$ and $N_{xy}$ defined at each Gaussian point.

$$\begin{cases} N_x = \sigma_{xx} h \\ N_y = \sigma_{yy} h \\ N_{xy} = \sigma_{xy} h \end{cases}$$

$\Longleftarrow$ with the stresses at the Gauss point yielding $N_x$, $N_y$, $N_{xy}$.

$$[\sigma] = [G]\{\varepsilon\}$$

$$[G] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \left\{ \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\}^T$$

$$[\sigma] = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$$

Displacement function $\omega = \sum_i \sum_j c_{ij} \xi^i \eta^j$

— BCs : e.g. for fixed edge on $\xi = -1$

$$\omega(\xi=-1) = \sum_j \eta^j \left( \sum_i c_{ij} (-1)^i \right) = 0$$

$$\& \quad \omega'(\xi=-1) = \sum_j \eta^j \left( \sum_{i-1} c_{ij} \cdot i \cdot (-1)^{i-1} \right) = 0$$

this is different than what's shown in the paper
I assume this is the correct BC.

Bending strain energy:

$$U = \frac{D}{2} \int_A \left\{ \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y} \right)^2 - 2(1-\nu) \left[ \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dA$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial \xi^2} \left( \frac{\partial \xi}{\partial x} \right)^2 + \frac{\partial^2 w}{\partial \eta^2} \left( \frac{\partial \eta}{\partial x} \right)^2 + \frac{\partial w}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial w}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2} + 2 \frac{\partial^2 w}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial \xi^2} \left( \frac{\partial \xi}{\partial y} \right)^2 + \frac{\partial^2 w}{\partial \eta^2} \left( \frac{\partial \eta}{\partial y} \right)^2 + \frac{\partial w}{\partial \eta} \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial w}{\partial \xi} \frac{\partial^2 \xi}{\partial y^2} + 2 \frac{\partial^2 w}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial \xi \partial \eta} \left( \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right) + \frac{\partial^2 w}{\partial \xi^2} \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + \frac{\partial^2 w}{\partial \eta^2} \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \xi} \frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial w}{\partial \eta} \frac{\partial^2 \eta}{\partial x \partial y}$$

Potential Energy with constraints:

$$\Pi = U - T + \lambda_i R_i$$

Minimization of $\Pi$ yields the critical Loads:

$$\frac{\partial \Pi}{\partial c_{ij}} = 0 \quad \text{and} \quad \frac{\partial \Pi}{\lambda_i} = 0$$

This gives the eigenvalue problem:

$$\left(\begin{bmatrix} \frac{\partial U}{\partial C_{ij}} & R_i^T \\ R_i & 0 \end{bmatrix} - P\begin{bmatrix} \frac{\partial T}{\partial C_{ij}} & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{Bmatrix} C_{ij} \\ \lambda_i \end{Bmatrix} = 0$$

$P$ : critical load factor

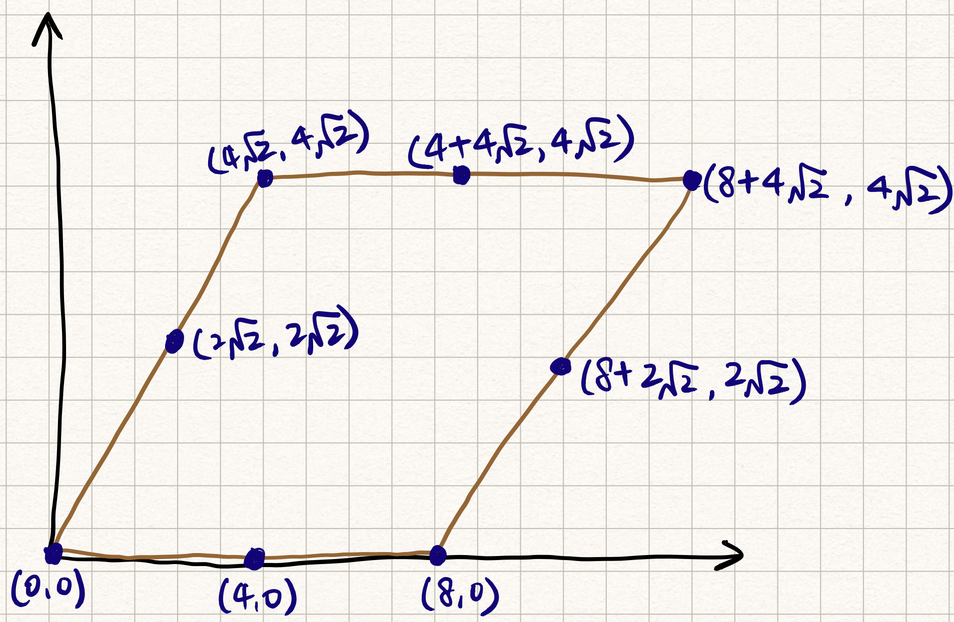
$P > 1$ : $P$ times that applied loads are necessary for the plate to buckle

$P < 1$ : the applied load will buckle the plate

$P = 1$ : the applied Load is the buckling Load

Example Case :

$a=8$ , simply-supported on all edges




Even this one only has 4 vertices, try to find 4 more points to make it 8 nodes so that it can be used in the equations derived before

Mapping function :

$$x = 4\xi + 2\sqrt{2}\eta + 4 + 2\sqrt{2}$$

$$y = 2\sqrt{2}(1+\eta)$$

Therefore : $|J| = 6\sqrt{2}$

Displacement for plane stress :

$$u = \sum_i \sum_j a_{ij} \xi^i \eta^j$$

$$v = \sum_i \sum_j b_{ij} \xi^i \eta^j$$